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Percolation processes in two dimensions II. Critical concentrations and the mean size index

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Abstract. New series data are examined for the mean cluster size for site and bond mixtures in two dimensions. The critical concentration for the site problem on the simple quadratic lattice is estimated as $p_c = 0.593 \pm 0.002$ and on the honeycomb lattice as $p_c = 0.698 \pm 0.003$. It is concluded that the data are reasonably consistent with the hypothesis that the mean cluster size $S(p) \simeq C(p_c - p)^{-\gamma}$ as $p \to p_c-$ with γ a dimensional invariant, $\gamma = 2.43 \pm 0.03$ in two dimensions. Estimates of the critical amplitude C are also given.

1. Introduction

In this paper we examine new series data for the mean cluster size for site and bond mixtures in two dimensions. We have introduced the problem, and given the new data, in a companion paper (Sykes and Glen 1976, to be referred to as I). In a preliminary investigation (Sykes *et al* 1973) it was concluded from an examination of the simple quadratic lattice that the critical concentration did not correspond to the radius of convergence of the series expansion for the mean cluster size; we have found that the same conclusion holds for both site and bond mixtures on the triangular, simple quadratic and honeycomb lattices and their corresponding matching lattices. Our objective is to estimate the critical concentration for those problems for which it is not known exactly and to investigate the hypothesis (Sykes and Essam 1964a) that the critical index for the mean cluster size is a dimensional invariant. Explicitly we investigate the hypothesis that

$$S(p) \simeq C(p_c - p)^{-\gamma}, \qquad p \to p_c- \tag{1.1}$$

for eight problems: the site and bond problems on the triangular, simple quadratic and honeycomb lattices, abbreviated as T(S), T(B), SQ(S), SQ(B), HC(S), HC(B) and the site problem on the simple quadratic matching lattice and the honeycomb matching lattice, abbreviated as SQM(S) and HCM(S) respectively. A detailed treatment of these matching lattices and the motivation for their study is given by Sykes and Essam (1964b) and Essam (1972). The critical concentration p_c is known exactly for the four problems:

$$\begin{array}{ll} \text{T(S)} & p_c = \tfrac{1}{2} \\ \text{SQ(B)} & p_c = \tfrac{1}{2} \\ \text{T(B)} & p_c = 2\sin(\pi/18) = 0.34729\ldots \\ \text{HC(B)} & p_c = 1 - 2\sin(\pi/18) = 0.65270\ldots. \end{array} \tag{1.2}$$

The critical concentrations for the matching pair SQ(S) and SQM(S) are complementary and likewise for the matching pair HC(S) and HCM(S).

Because of the relevance of the critical index γ to theories of scaling we have made an extensive study of the data (see Gaunt and Guttmann 1974 for a review of extrapolation procedures in general); however we have found it very difficult to draw precise conclusions. We have therefore confined our present account to a brief summary of the standard Padé approximant analysis which in our view is at least as good as any alternative procedure.

2. Padé approximant analysis

To study the expansions for $S(p)$ given in I (tables 2 and 3) we have followed the procedure described in detail by Gaunt and Guttmann (1974). On forming Padé approximants to the series for $(d/dp) \ln S(p)$ it is found that the number and location of singularities inside or on the circle $|p| = p_c$ varies widely from problem to problem. However, in all cases, it appears that the closest singularity is not at p_c but lies in the left half of the p plane, sometimes on the real axis, sometimes in the complex plane. We omit the details which do not appear especially significant. The essential point is that we are faced with the situation of a strong physical singularity at p_c which is dominated asymptotically by one or more weak non-physical singularities closer to the origin. In such circumstances we should not anticipate the convergence of the approximants in the vicinity of p_c to be rapid. In practice we have found that convergence tends to be poorest for those problems for which the exact value of p_c is unknown. The reason for this is not understood but appears to be unrelated to the behaviour of the non-physical singularities which seem comparable for both site and bond problems.

We give in tables 1–4 several sequences of Dlog Padé estimates of p_c and γ (given by the poles and residues respectively) for those problems for which p_c is known exactly. Although the last few estimates are reasonably close to the exact value of p_c in all cases, the sequences all exhibit small irregularities with no definite trend (see Gaunt and Guttmann 1974 for a discussion of this phenomenon). If for each problem the residues are plotted against the position of their corresponding poles the last few estimates are found to define quite accurately a single smooth curve irrespective of which of the three sequences they come from. The residue which would be obtained if a pole were located exactly at p_c can

Table 1. Dlog Padé estimates of p_c (and γ) for the honeycomb bond problem.

n	$[n-1/n]$	$[n/n]$	$[n+1/n]$
3	0·6478 (1·869)	0·6000 (1·392)	0·5026 (0·567)
4	0·6710 (2·025)†	0·5684 (1·104)‡	none
5	0·6075 (1·523)	0·6218 (1·714)	0·6430 (2·104)
6	0·5694 (1·485)†	0·6489 (2·253)	0·6445 (2·136)
7	0·6356 (1·915)	0·6439 (2·120)	0·6444 (2·134)‡‡
8	0·6505 (2·337)	0·6657 (3·385)	0·6544 (2·500)
9	0·6562 (2·605)		

† Defect on positive axis.
‡ Defect on negative axis.

Table 2. Dlog Padé estimates of p_c (and γ) for the simple quadratic bond problem.

n	$[n-1/n]$	$[n/n]$	$[n+1/n]$
2	0·4082 (1·225)	0·6124 (4·134)	0·1862 (0·020)
3	0·4564 (1·584)	0·4686 (1·740)	0·4879 (2·087)
4	0·4207 (1·750)†	0·5086 (2·736)	0·4941 (2·226)
5	0·4820 (1·897)	0·4981 (2·352)	0·5206 (4·740)†
6	0·5022 (2·518)	0·5012 (2·473)	0·5036 (2·601)
7	0·5020 (2·507)‡		

† Defect on positive axis.
‡ Defect on negative axis.

Table 3. Dlog Padé estimates of p_c (and γ) for the triangular bond problem.

n	$[n-1/n]$	$[n/n]$	$[n+1/n]$
2	0·3074 (1·571)	0·3207 (1·767)	0·3430 (2·272)
3	none	0·3479 (2·440)	0·3445 (2·316)
4	0·3394 (2·121)	0·3459 (2·366)‡	0·3435 (2·290)†‡
5	0·3521 (2·684)		

† Defect on positive axis.
‡ Defect on negative axis.

Table 4. Dlog Padé estimates of p_c (and γ) for the triangular site problem.

n	$[n-1/n]$	$[n/n]$	$[n+1/n]$
2	0·4082 (1·225)	0·6124 (4·134)	0·2131 (0·037)
3	0·4622 (1·638)	0·4736 (1·788)	0·4837 (1·969)
4	none	0·4384 (1·951)†	0·4960 (2·254)
5	none	0·5007 (2·422)	0·4982 (2·323)
6	0·4956 (2·219)	0·4965 (2·256)	0·4979 (2·314)‖
7	0·4955 (2·215)†	0·4992 (2·368)‖	

† Defect on positive axis.
‖ Defect in complex plane.

be read off from the plot and in this way we estimate

$$\begin{aligned} \gamma &= 2{\cdot}434 \pm 0{\cdot}009 \qquad \text{HC(B)} \\ \gamma &= 2{\cdot}425 \pm 0{\cdot}005 \qquad \text{SQ(B)} \\ \gamma &= 2{\cdot}423 \pm 0{\cdot}008 \qquad \text{T(B)} \\ \gamma &= 2{\cdot}40 \pm 0{\cdot}03 \qquad \text{T(S)}. \end{aligned} \tag{2.1}$$

The large uncertainty for the triangular site problem arises because there is only one point with $p > p_c$ and we are reluctant to attach much weight to it.

For the remaining four problems, all site problems for which the p_c are not known exactly, the Dlog Padé approximants yield sequences which are not particularly well converged and it is not possible to make precise estimates of p_c and γ. However if we make the assumption (not unreasonable in view of (2.1)) that matching pairs have the same value of γ then this value, and the critical concentrations will correspond to the intersection of the pole–residue plot of each problem with the pole–residue plot of the matching problem on a complementary probability scale.

The resulting graphs for the honeycomb and simple quadratic matching pairs of lattices are given in figures 1 and 2 respectively. Unfortunately the plots only intersect after extrapolation, implying that all the usable estimates of p_c are below the true value for each lattice. Since the degree of curvature of the plots in the vicinity of p_c is not known, extrapolation necessarily increases the uncertainties, particularly in the value of γ. Nevertheless we feel the figures justify the estimates:

$$\begin{aligned} \gamma &= 2.41 \pm 0.04 \qquad \text{HC(S), HCM(S)} \\ \gamma &= 2.40 \pm 0.06 \qquad \text{SQ(S), SQM(S)} \end{aligned} \tag{2.2}$$

in reasonably good agreement with (2.1). Because of the apparent direction of the curvature, estimates of the critical probability can be made more precisely, namely:

$$\begin{aligned} p_c &= 0.698 \pm 0.003 \qquad \text{HC(S)} \\ p_c &= 0.302 \pm 0.003 \qquad \text{HCM(S)} \\ p_c &= 0.593 \pm 0.002 \qquad \text{SQ(S)} \\ p_c &= 0.407 \pm 0.002 \qquad \text{SQM(S).} \end{aligned} \tag{2.3}$$

Various methods of improving the estimates (2.3) have been tried but without success and accordingly we adopt them as our final estimates. Likewise we have been unable to improve on the estimates of the critical exponent γ, although ample supportative evidence has been found. For example, evaluation of Padé approximants to the $(p_c - p)(\mathrm{d}/\mathrm{d}p) \ln S(p)$ series at $p = p_c$ gives estimates of γ which are consistent with (2.1) and (2.2). The results for the honeycomb bond problem are given in table 5 and are typical of those found for

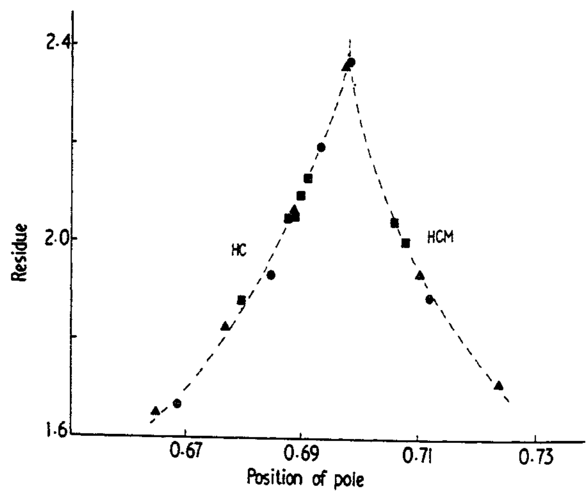

Figure 1. Pole–residue plot for the site problem on the honeycomb lattice (against p) and the honeycomb matching lattice (against $1-p$). ■, $[n/n]$; ▲, $[n-1/n]$; ●, $[n+1/n]$.

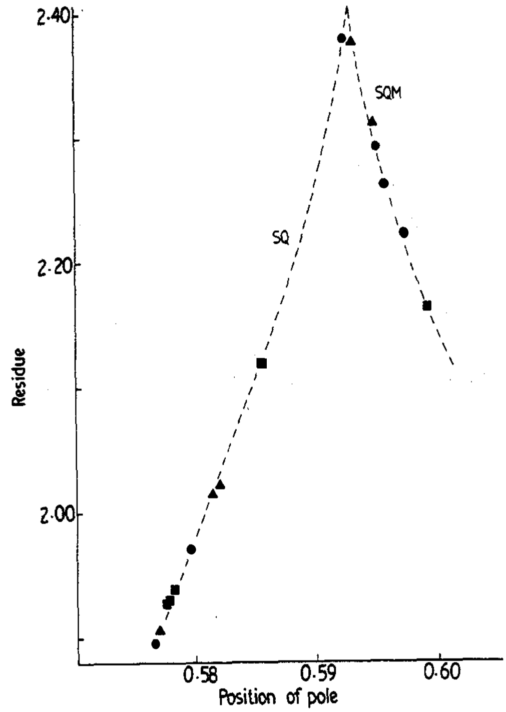

Figure 2. Pole–residue plot for the site problem on the simple quadratic lattice (against p) and the simple quadratic matching lattice (against $1-p$). ■, $[n/n]$; ▲, $[n-1/n]$; ●, $[n+1/n]$.

Table 5. Padé estimates of γ for the honeycomb bond problem using the $(p_c-p)(d/dp)\ln S(p)$ series and the exact value of p_c.

n	$[n-1/n]$	$[n/n]$	$[n+1/n]$
1	1·3054	1·6453	2·2245
2	none	1·9081	1·9223
3	1·9231	1·9064†	1·8664†
4	1·8707†	1·9112‖	2·2062
5	2·8854	2·5986	2·3587
6	3·1342†	2·3735	2·3558†
7	2·4539	2·4236	2·4349
8	2·4310	2·4279	2·4096
9	2·4358†		

† Defect on positive axis.
‖ Defect in complex plane.

the problems with exactly known p_c. For the remaining problems the results are less precise because of the uncertainty in p_c and somewhat poorer convergence but are in general accord with (2.2).

The estimates (2.3) are close to, and certainly well within the uncertainties of the much earlier, though less precise, estimates of Sykes and Essam (1964a) based on shorter series. From the estimates (2.1) and (2.2) it seems likely that γ is lattice independent with a value around

$$\gamma = 2{\cdot}43 \pm 0{\cdot}03. \tag{2.4}$$

As is usual in problems of this kind the uncertainties are not strict error bounds, but just represent a subjective assessment of the rate of convergence of the available numerical data (see for example Gaunt and Guttmann 1974). The estimate (2.4) is only 2% larger than the earlier estimate of Sykes and Essam (1964a) although it lies just outside their uncertainty limits.

We have used the exact or best estimates of p_c together with (2.4) to estimate the critical amplitudes C defined by (1.1). We have used two methods: the calculation of the residues at the pole close to p_c of the Padé approximants to the series for $[S(p)]^{1/\gamma}$ and also evaluation of the Padé approximants to the series for $(p_c - p)[S(p)]^{1/\gamma}$ at $p = p_c$. Results obtained by these methods are in excellent agreement leading to final estimates for C of

$$\begin{array}{ll} 0{\cdot}145 \pm 0{\cdot}001 & \textsc{hc(b)} \\ 0{\cdot}134 \pm 0{\cdot}001 & \textsc{sq(b)} \\ 0{\cdot}084 \pm 0{\cdot}001 & \textsc{t(b)} \end{array} \tag{2.5}$$

and

$$\begin{array}{ll} 0{\cdot}140 \pm 0{\cdot}006 & \textsc{hc(s)} \\ 0{\cdot}147 \pm 0{\cdot}003 & \textsc{sq(s)} \\ 0{\cdot}128 \pm 0{\cdot}003 & \textsc{t(s)} \\ 0{\cdot}104 \pm 0{\cdot}006 & \textsc{sqm(s)} \\ 0{\cdot}064 \pm 0{\cdot}003 & \textsc{hcm(s)}. \end{array} \tag{2.6}$$

The uncertainties in p_c (where applicable) and in γ each introduce additional uncertainties in C of the same order as those quoted in (2.5) and (2.6). For the bond problem the amplitudes are seen to decrease monotonically with increasing lattice coordination number. The same is probably true of the site problem and is within the range of uncertainties for the honeycomb and simple quadratic lattices. Such behaviour is in agreement with the Bethe approximation (Fisher and Essam 1961). However unlike the Bethe approximation it does not seem that on a given lattice the amplitude for the bond problem is always greater than for the corresponding site problem.

3. Conclusions

All the available series data have been found reasonably consistent with the hypothesis that the mean size index γ is a dimensional invariant in two dimensions for both site and bond problems. (The equivalence of γ for these two problems then follows from the argument of I, § 3.) We have found it difficult to draw precise conclusions. The methods described in I could be used to add a further coefficient or two in all cases but we have not thought this worthwhile because of the poor convergence already experienced. Our final estimate of $\gamma = 2{\cdot}43 \pm 0{\cdot}03$ is close to $2\frac{3}{7} = 2{\cdot}428\ldots$ and we adopt this simple fraction as a convenient mnemonic to replace the earlier tentative value of $2\frac{3}{8}$ of Sykes and Essam (1964a).

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